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The diffraction of SH waves by an arbitrary shaped crack in two dimensions

By P. A. Lewis and G. R. Wickham
Department of Mathematics, University of Manchester, Manchester M13 9PL, U.K.

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In this paper we consider the two-dimensional scalar scattering problem for Helmholtz's equation exterior to a smooth open arc of general shape. The problem has a number of physical applications including the diffraction of sound by a rigid barrier immersed in a compressible fluid and by a crack in an elastic solid which supports a state of anti-plane strain (SH-motion). The mathematical method used here is the crack Green function method introduced by G. R. Wickham. This enables the scattering problem to be reduced to the solution of a Fredholm integral equation of the second kind with a continuous kernel. The numerical solution of this equation is discussed and a number of examples are computed.

1. Introduction

The diffraction of waves by a thin planar strip in two dimensions has probably received more attention than any other classical scattering problem for the scalar Helmholtz equation. There are a number of reasons for this, both physical and mathematical. From the former point of view, it occurs in various branches of

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classical physics, particularly in acoustics, electromagnetic theory and elastodynamics, while in the context of mathematical methods, it provides an ideal benchmark problem for testing new analytical and numerical techniques. The important physical characteristic of the problem is that it is the simplest finite geometry which gives rise to multiple edge diffraction. This in turn has mathematical implications in that the edges of the strip are singular points of the solution of the scattering boundary-value problem. It follows that, if the problem is to be solved numerically at some stage, then the numerical method should be designed accordingly.

Our interest in the corresponding problem when the strip is *non-planar* arises in the context of linear elastodynamics. The exterior Neumann boundary-value problem models the scattering of SH-waves in anti-plane strain by a strip-like crack in a homogeneous isotropic elastic solid. In itself this problem is of little physical interest but it may be regarded as a prototype of the important case of plane strain where the wave motion consists of longitudinal (P-waves) and vertically polarized shear waves (SV-waves), coupled at the crack faces. The mathematical theory that we shall present here is easily generalized to the latter case and will be the subject of a following paper (Lewis 1992). The main justification for the present work, quite apart from the fact that our solution is new, is to present a mathematical method in an environment uncluttered by algebraic complexity. Our aim is to improve on methods for quantifying the diffraction of elastic waves by cracks, a subject which could be said to have played a pivotal role in the understanding of the reliability of ultrasonic methods for detecting metallurgical defects in crucial components of plant and machinery. However, it should not escape the reader's attention that what we have to say here will be of use in various other branches of continuum mechanics.

Most of the mathematical methods used for the planar case are, in some sense, special. For example, it is possible to regard the strip as the limiting case of an ellipse with unit eccentricity. The wave equation is separable in elliptic coordinates and so it is possible to obtain an explicit infinite series solution (Harumi 1961). Similarly, the scattering problem may be formulated as a mixed boundary value problem for a semi-space in a way which is suitable for the application of Fourier transform techniques (Ang & Knopoff 1964*a*, *b*; Sih & Loeber 1968, 1969; Mal 1970). The latter are just a few examples where elastodynamics has provided the motivation. Generally, they are formulations that provide an exact analytical approach to low-frequency asymptotic expansions and also yield Fredholm integral equations which are easy to solve when the wavelength of the incident waves is comparable with the dimensions of the crack. For high frequencies Keogh (1985*a*, *b*) has provided an exact convergent series for diffraction by a planar strip-like crack which has the powerful feature that it provides a rigorous asymptotic expansion for the diffracted field to any desired order of approximation. Keogh used the Wiener–Hopf technique and thereby generalized to the elastic case the scalar solutions obtained by a number of other authors (see, for example, Karp & Russek 1956; Wolfe 1972).

Methods which exploit the fact that the crack faces are flat are not easily generalized to arbitrary non-planar geometries. In this case it is natural to use a Green function to derive an integral equation on the scattering surface. This in turn may be treated using a suitable numerical scheme such as the boundary element method. In the applications to scattering by a crack, it may be shown that, in general, the solution may be determined in terms of the crack opening displacement (COD). Thus, for example, the problem may be formulated in terms of a certain integro-differential operator equation with the COD as the unknown function. A particular feature of this type of equation is that it is hyper-singular (Martin & Rizzo 1989). One way over the inevitable numerical difficulties with such an equation is to regularize the operator by performing an integration by parts. This essentially reformulates the equation in terms of a new unknown function which is a tangential derivative of the COD. The latter may be interpreted physically in terms of an equivalent dislocation density for the crack. There are a number of numerical approaches to the regularized equation. In particular, Tan (1977) used a Fourier expansion for the unknown dislocation densities but this failed to take proper account of the singularities at the edge of the crack. In contrast, the method first proposed by Erdogan & Gupta (1972) takes explicit account of these singularities. This has proved to be a highly successful approach to solving the hyper-singular equation for flat cracks (see, for example, Brind & Achenbach 1981). Slightly different, but related methods were given by van den Berg (1982) and Takakuda (1983) who also confined their attention to the planar case.

Martin & Rizzo (1989) proposed a direct boundary element attack on the hyper-singular equation. They argued that 'the strategy of regularisation' is '... a burdensome and conservative posture'. Their approach is to introduce boundary elements that allow the evaluation of the hyper-singular operator interpreted as a Hadamard finite part. Thus, in particular, such a scheme must take explicit account of the singularities in the COD and the finite elements must be chosen to aid the calculation of the finite part integrals. This of course is itself an implicit regularization. Thus we are presented with yet another example of a commonly experienced heuristic principle which may be termed 'the principle of conservation of mathematical difficulty'! If there is a point of contention, then it may be encapsulated in the question: Is it preferable to perform an analytical or a numerical regularization of the governing equations? We take an opposite stance to Martin & Rizzo, namely that, in general, it is better to prepare the conditioning of the equation to be solved by analytical rather than numerical methods. Indeed Wickham (1981) argued that the real cause of the difficulty in solving this type of problem is that the free space Green function is an inappropriate fundamental solution for building a representation of the scattered field. He showed that for the planar crack it was possible to construct a crack Green function (CGF) which is discontinuous across the crack surface and when allied with the scattered field in Green's theorem yields a Fredholm integral equation of the second kind for the COD. This integral equation has the endearing feature that the singularities in the COD are *explicitly* displayed and the kernel is completely continuous. Further, this simple structure has an obvious pay-off in the construction of a numerical scheme; the numerical solution of Fredholm second kind equations is extremely well understood and documented. Indeed, there are numerous successful schemes to choose from (Baker 1977). Of course, by the principle already enunciated, there must be a price to pay in Wickham's approach. In fact, it turns out that all the numerical difficulty in the problem has been transferred into the evaluation of the kernel. The latter is represented by a weakly singular integral operator acting on a known continuous function. Wickham (1982) argued that the cost may simply be met by a suitable product integration scheme. Further, given that the kernel is explicitly known, it is relatively straight forward to control the accuracy of the computation.

In this paper we demonstrate that the CGF method has analytical and numerical advantages over other regularization techniques. It focuses attention on the 'real' unknown COD rather than the more singular dislocation density and the essential structure of the solution is explicitly displayed in the Fredholm second kind equation. The latter affords a flexible numerical approach in which the errors may be rigorously controlled. Above all, the CGF method provides a very simple extension to non-planar geometries without resorting to the direct numerical treatment of the hyper-singular equation. In §2 we formulate the exterior Neumann scattering problem for a finite open arc and prove that under certain edge and radiation conditions it has a unique solution. In §3 we explore the classical double layer solution for the scattered field as an integral over the arc of the COD times a dipole corresponding to the free space Green function and in §4 we introduce the CGF that enable the determination of this dipole distribution as the solution of a Fredholm integral equation of the second kind. The remainder of the paper is devoted to the numerical treatment of a number of simple examples; we reproduce some of the known results for the planar case and some new results for a variety of non-planar geometries. In a subsequent paper (Lewis 1992) a generalization of the present theory to the full equations of elasticity in two dimensions will be given.

2. Formulation of the scattering problem

Let L be a simple open and bounded arc in an infinite plane $-\infty < x < \infty$, $-\infty < y < \infty$, where (x, y) are cartesian coordinates relative to a fixed origin O, and let D_0 be the domain exterior to L. Suppose that the points on L may be parametrized according to

$$x = f(s), \quad y = g(s),$$

where s is arc length measured from one of the ends and assume that f and g have continuous second derivatives. We shall label the two sides of the arc by L_+ and L_-, where L_+ is the left-hand side as one moves in the direction of increasing arc length, and the two ends of L shall be denoted by $E^{\pm}$, where E^+ and E^- correspond to $s = \pm s_0$ respectively (see figure 1). The unit normal to $L_{\pm}$ pointing into the domain D_0 will be denoted by $\pm \boldsymbol{n}(s)$ and if $P, Q, \ldots,$ denote points in $D_0 \cup L$, then our scattering problem may be stated as follows.

Problem 1. Scattering problem $S(\phi_0)$. Determine a function $\phi(P)$ such that $\phi(P) \in C^2(D_0)$ and whose first-order derivatives are continuous from $L_{\pm}$ and which satisfies the Helmholtz equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)\phi(P) = 0, \quad P \in D_0, \tag{2.1}$$

where k is a non-zero complex constant such that $0 \leqslant \arg(k) \leqslant \frac{1}{2}\pi$; the boundary condition

$$\frac{\partial \phi}{\partial n}(P) + \frac{\partial \phi_0}{\partial n}(P) = 0, \quad P \in L_{\pm}, \tag{2.2}$$

where $\phi_0(P)$ is any solution of (2.1) such that $\partial\phi_0/\partial n$ is Holder continuous on L; the 'edge condition', namely there exists $M > 0$ such that

$$|\phi(P)| \leqslant M, \quad P \in D_0; \tag{2.3}$$

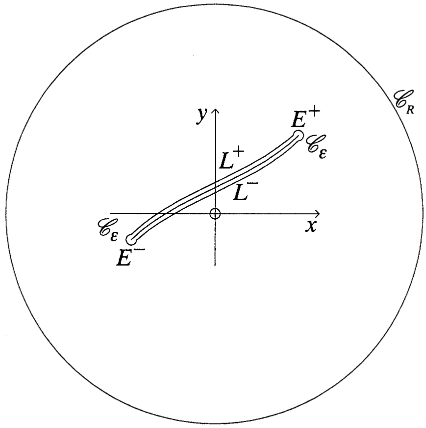

Figure 1. Configuration of the scattering problem.

and the radiation condition

$$r^{\frac{1}{2}}(\partial\phi/\partial r - \mathrm{i}k\phi) \to 0, \tag{2.4}$$

as $r \to \infty$, uniformly in the angle θ, where (r, θ) are plane polar coordinates defined by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad 0 \leqslant \theta < 2\pi. \tag{2.5}$$

(Wilcox (1959) has shown that (2.4) is equivalent to the usual Sommerfeld conditions.)

Theorem 1. *The only solution of* $\mathscr{S}(0)$ *is identically zero.*

The proof of this result is given in Appendix A. In the remainder of this paper we shall construct the exact solution of $\mathscr{S}(\phi_0)$ and compute it in some special cases.

3. The double layer solution

We shall seek a solution of the scattering problem in the form of the double layer potential

$$\phi(P) = \int_L \mu(s_q) \frac{\partial}{\partial n_q} G(P, q)\, \mathrm{d}s_q, \quad P \in D_0, \tag{3.1}$$

where, here, lower case letters shall denote points on L,

$$G(P, Q) = \tfrac{1}{4}\mathrm{i}H_0^{(1)}(k|\boldsymbol{r}(P) - \boldsymbol{r}(Q)|) \tag{3.2}$$

and $\boldsymbol{r}(P)$ denotes the position vector of P relative to O. We shall assume that the density function $\mu(q)$ satisfies

Properties $\mathscr{P}(L)$

$$\mu(E^-) = 0, \quad \mu(E^+) = 0,$$

$$\mathrm{d}\mu/\mathrm{d}s_q \quad \text{exists and is Holder continuous on } L.$$

In Appendix B we prove the following.

Theorem 2. *The integral representation* (3.1) *solves the scattering problem* $\mathscr{S}(\phi_0)$ *provided we can determine a function* $\mu(s_q)\in\mathscr{P}(L)$ *such that*

$$-\frac{\partial\phi_0}{\partial n_p}=\frac{\partial}{\partial n_p}\int_L \mu(s_q)\frac{\partial G}{\partial n_q}(p,q)\,\mathrm{d}s_q,\quad p\in L^{\pm}. \tag{3.3}$$

Equation (3.3) is a formal statement of the boundary conditions (2.2) and is the hyper-singular equation discussed by Martin & Rizzo (1989).

4. Green functions in D_0

We now introduce a class of Green functions in the plane cut along L. First, we define $\hat{G}(P,q)$ by

$$\hat{G}(P,q)=\int_L \rho(t,q)\frac{\partial G}{\partial n_t}(t,P)\,\mathrm{d}s_t,\quad P\in D_0, \tag{4.1}$$

where

$$\rho(t,q)=-(2/\pi)\{\ln|s_t-s_q|-\ln|s_0^2-s_q s_t-\surd(s_0^2-s_t^2)\surd(s_0^2-s_q^2)|+\ln|s_0|\}, \tag{4.2}$$

where the square roots in this expression are taken to be positive. Now, it is clear that ix$P\equiv(x,y)$, then

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k^2\right)\hat{G}(P,q)=0,\quad P\in D_0$$

and $\hat{G}$ satisfies the radiation condition (2.4). Further, we may express $\hat{G}$ in the form

$$\hat{G}(P,q)=\mathrm{Re}\left[\frac{1}{2\pi\mathrm{i}}\int_L\frac{\rho(t,q)}{(t-z)}\mathrm{d}t\right]+\hat{G}_1(P,q),$$

where $z=x+\mathrm{i}y$ and

$$\hat{G}_1(P,q)=\int_L \rho(t,q)\frac{\partial G_1}{\partial n_t}(t,P)\,\mathrm{d}s_t \tag{4.3}$$

is continuous in the whole plane $D_0\cup L$. Thus, provided $p\neq q$

$$[\hat{G}(p,q)]=\hat{G}_+(p,q)-\hat{G}_-(p,q)=\rho(p,q),\quad q\in L. \tag{4.4}$$

We conclude that $\hat{G}(p,q)$ is a single valued outgoing solution oxHelmholtz's equation in the plane cut along L and has an isolated singularity at $P=q$. In Appendix C we prove the following.

Theorem 3. (*a*) *Let* $z=q+R\,\mathrm{e}^{\mathrm{i}\Theta}$, *then as* $z\to q$ *from the* $+$ *or* $-$ *sides of* L,

$$\frac{\partial\hat{G}}{\partial R}=\mp\frac{1}{\pi R}+O(1). \tag{4.5}$$

(*b*) *If* $t-p=R(t,p)\,\mathrm{e}^{\mathrm{i}\Theta(t,p)}$ *then*

$$\partial\hat{G}/\partial n_p=\hat{K}(p,q),\quad p\in L,\quad q\in L,\quad p\neq q, \tag{4.6}$$

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where
$$\hat{K}(p,q) = \hat{K}_0(p,q) + \int_L \rho(t,q) M_1(t,p)\,\mathrm{d}s_t, \tag{4.7}$$

$$\hat{K}_0(p,q) = \frac{1}{2\pi}\frac{\partial}{\partial n_p}\int_L \rho(t,q)\frac{\partial \Theta}{\partial s_t}(t,p)\,\mathrm{d}s_t \tag{4.8}$$

$$= \frac{1}{2\pi}\frac{\partial}{\partial n_p}\int_L \rho(t,q)\frac{\sin\alpha(t,p)}{R(t,p)}\,\mathrm{d}s_t, \tag{4.9}$$

$$M_1(t,p) = \frac{\partial^2 G_1}{\partial n_p\,\partial n_t}(t,p), \tag{4.10}$$

and $\hat{K}(p,q) - \hat{K}_0(p,q)$ *is continuous for all* p, q *on* L *and* $\hat{K}_0(p,q) = O(\ln|p-q|)$.

This result shows that $\hat{G}(P,q)$ has equal and opposite source singularities at $P = q^+$ and $P = q^-$ and has a continuous normal derivative on L except possibly at q. Accordingly we shall call any function with these properties a 'Green function for (2.1) in the cut plane D_0'. It is clear that there are infinitely many Green functions in D_0 for suppose $\sigma(t,q)$ is any function defined on $L \times L$ such that

$$\sigma(t,q) \in \mathscr{P}_t(L) \times \mathscr{P}_q(L),$$

i.e. for each q, σ has properties $\mathscr{P}_t(L)$ and for each t, σ has properties $\mathscr{P}_q(L)$, then the function G_σ defined by

$$G_\sigma(P,q) = \int_L \sigma(t,q)\frac{\partial G}{\partial n_t}(t,P)\,\mathrm{d}s_t \tag{4.11}$$

is, by Theorem 3, an outgoing solution of the wave equation satisfying (2.3) and having a continuous normal derivative on L. The latter statement holds for each value of $q \in L$ and hence

$$G^*(P,q) = \hat{G}(P,q) + G_\sigma(P,q) \tag{4.12}$$

is a Green function in D_0. To this extent we may regard $\hat{G}(P,q)$ as the 'fundamental Green function' for equation (2.1) in D_0. In the following section we shall show that the exact solution of the scattering problem $\mathscr{S}(\phi_0)$ may be calculated in terms of any function of the form (4.12).

5. Existence theory

We shall now apply Green's theorem to the functions $\hat{G}(P,q)$ and $\phi(P)$ in the domain $\mathscr{D}_\epsilon$ described in figure 1 excluding small semi-circular areas around $P = q^+$ and $P = q^-$; i.e.

$$\int_{\mathscr{C}}\left(\phi\frac{\partial \hat{G}}{\partial n_p} - \hat{G}\frac{\partial \phi}{\partial n_p}\right)\mathrm{d}s_p = 0. \tag{5.1}$$

Assuming for the present that the contribution from $\mathscr{C}_\epsilon$ vanishes in the limit $\epsilon \to 0$ and using theorem 3 and (2.2) we find that as $\delta \to 0$ and $R \to \infty$

$$\mu(q) - \int_L \hat{K}(p,q)\,\mu(p)\,\mathrm{d}s_p = \int_L \rho(t,q)\frac{\partial \phi_0}{\partial n_t}\,\mathrm{d}s_t, \tag{5.2}$$

where
$$\mu(q) = [\phi(q)], \quad q \in L.$$

Similarly, the same process applied to $G_\sigma(P,q)$ and $\phi(P)$ gives

$$-\int_L K_\sigma(p,q)\,\mu(p)\,\mathrm{d}s_p = \int_L \sigma(t,q)\frac{\partial\phi_0}{\partial n_t}\mathrm{d}s_t, \quad q\in L, \tag{5.3}$$

which when added to (5.2) gives

$$\mu(q) - \int_L K^*(p,q)\,\mu(p)\,\mathrm{d}s_p = \int_L \rho^*(t,q)\frac{\partial\phi_0}{\partial n_t}\mathrm{d}s_t, \quad q\in L, \tag{5.4}$$

where
$$K^*(p,q) = \hat{K}(p,q) + K_\sigma(p,q) \tag{5.5}$$
and
$$\rho^*(t,q) = \rho(t,q) + \sigma(t,q). \tag{5.6}$$

Thus, assuming a solution $\mathscr{S}(\phi_0)$ exists such that

$$\lim_{\epsilon\to 0}\int_{\mathscr{C}_\epsilon} \ldots = 0$$

in (5.1), then the potential difference $[\phi]$ across L satisfies any one of the integral equations (5.4). Each of these equations is equivalent to the fundamental equation (5.2) in the sense that any solution of one is also a solution of (5.4) and vice versa. Further if $\partial\phi_0/\partial n_t$ is sufficiently smooth the right-hand sides of (5.4) and (5.2) are continuous and, since the kernels $\hat{K}$ and K^* have at worst a logarithmic singularity at $p = q$, then the Fredholm alternatives (Smithies 1962) apply. The main result of this section is the following.

Theorem 4. *There exists one and only one solution of the scattering problem* $\mathscr{S}(\phi_0)$ *given by* (3.1), *where* $\mu(q)$ *is the unique solution of any one of the integral equations* (5.4).

The details of this proof are given in Appendix D. Thus we may compute $[\phi]$ by solving any of the integral equations (5.4). In the following sections we shall exploit this property to solve the scattering problem for a number of special cases.

6. Some examples; the straight line and the circular arc

For this, and the remaining sections, we introduce rectangular cartesian coordinates (x,y), such that the origin of the coordinate system is located at the midpoint of the straight line segment joining the endpoints, E^- and E^+, of the scatterer, and E^- and E^+ lie on the x axis. Thus E^- and E^+ are located at $(\pm a, 0)$ say. It is convenient to introduce the dimensionless parameter $N = ka$ and non-dimensionalize all the lengths in the problem, so that E^- and E^+ correspond to $(\pm 1, 0)$. Next we specialize the general integral equation (5.4) to two particular geometries.

(a) *The straight line segment*

The simplest case is when the scatterer is a straight line segment. Evidently (4.9) and (C 14) give

$$\hat{K}_0(p,q) \equiv 0, \tag{6.1}$$

since $\beta(t) = \Theta(t,p)\;\forall p\in L$. Equation (5.2) therefore becomes

$$\mu(x_q) - \int_{-1}^{1}\mu(x_p)\,\hat{K}_1(x_p,x_q)\,\mathrm{d}x_p = \int_{-1}^{1}\rho(x_t,x_q)\frac{\partial\phi_0}{\partial y_t}(x_t,0)\,\mathrm{d}x_t, \tag{6.2}$$

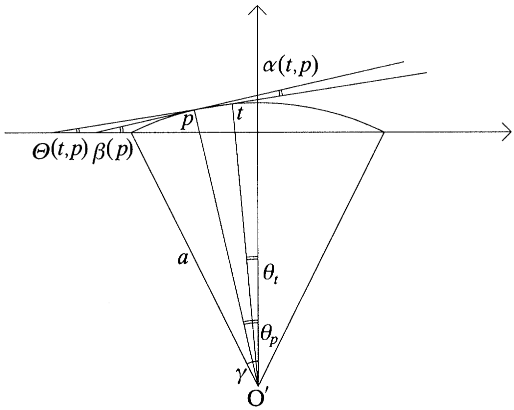

Figure 2. Circular scatterer geometry.

where $$\hat{K}_1(x_p, x_q) = \int_{-1}^{1} \rho(x_t, x_q) M_1(x_t, x_p)\,\mathrm{d}x_t$$

$$= \tfrac{1}{4}\mathrm{i}N^2 \int_{-1}^{1} \rho(x_t, x_q) \left\{ \frac{H_1^{(1)}(N|x_t - x_p|)}{N|x_t - x_p|} + \frac{2\mathrm{i}}{\pi N^2 (x_t - x_p)^2} \right\} \mathrm{d}x_t \tag{6.3}$$

and $$\rho(x_t, x_q) = -(2/\pi)\,[\ln|x_t - x_q| - \ln|1 - x_t x_q + \surd(1 - x_t^2)\,\surd(1 - x_q^2)|]. \tag{6.4}$$

It follows from (6.3) that $\hat{K}_1(x_p, x_q)$ is uniformly $O(N^2 \ln N)$ and is a continuous kernel. Thus the integral operator is compact and small in the limit $N \to 0$ and (6.2) may be written in the form

$$\mu(x_q) = \mu_0(x_q) + \int_{-1}^{1} \mu(x_p)\,\hat{K}_1(x_p, x_q)\,\mathrm{d}x_p. \tag{6.5}$$

It follows that (6.5) may be solved by iteration, the solution being expressed as a Liouville–Neumann series. In the case when the incident field is

$$\phi_0 = \mathrm{e}^{\mathrm{i}Ny}, \tag{6.6}$$

we find that, to leading order,

$$\mu(x_q) = -2\mathrm{i}N\,\surd(1 - x_q^2) + N^3\{\tfrac{1}{2}\pi\,\surd(1 - x_q^2)\,[\tfrac{1}{2} + (\mathrm{i}/\pi)\,\{\gamma - \tfrac{3}{2} + \ln(\tfrac{1}{4}N)\}]$$
$$+ \mathrm{i}[\tfrac{1}{3}(1 - x_q^2)^{\frac{3}{2}} + \tfrac{1}{2}x_q^2\,\surd(1 - x_q^2)]\} + O(N^5 \ln N). \tag{6.7}$$

This result is a useful asymptotic limit against which the numerical methods described later may be compared.

(*b*) *The circular arc*

Consider a scatterer in the form of an arc of a circle of radius a and centre O′ and which subtends an angle 2γ at O′, see figure 2. By using equations (4.9) and (C 14) we have

$$\hat{K}_0(p, q) = \frac{1}{2\pi} \frac{\partial}{\partial n_p} \int_L \rho(t, q) \frac{\sin\alpha(t, p)}{R(t, p)}\,\mathrm{d}s_t.$$

But, $$\frac{\sin\alpha(t, p)}{R(t, p)} = \frac{1}{2a}$$

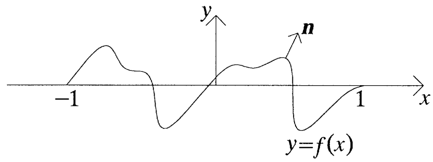

Figure 3. Geometry of the general profile.

and so $\hat{K}_0(p,q) \equiv 0$, as in the previous example. We next determine $M_1(t,p)$ in terms of the polar coordinates (r,θ) relative to the origin O′. It can be easily shown that

$$M_1(\theta_p,\theta_t) = \tfrac{1}{4}\mathrm{i}N^2\left[\frac{H_1^{(1)}(Nv)}{Nv} - \tfrac{1}{4}v^2 H_0^{(1)}(Nv) + \frac{2\mathrm{i}}{\pi N^2 v^2}\right], \tag{6.8}$$

where $N = ka$ and

$$v = 2\sin\tfrac{1}{2}|\theta_p - \theta_t|. \tag{6.9}$$

Finally, for this example, $\rho(\theta_t,\theta_q)$ is given by

$$\rho(\theta_t,\theta_q) = -(2/\pi)\,[\ln|\theta_t-\theta_q| - \ln|\gamma_0^2 - \theta_q\theta_t - \surd(\gamma_0^2-\theta_q^2)\,\surd(\gamma_0^2-\theta_t^2)| + \ln|\gamma_0|]. \tag{6.10}$$

Again we see that the operator is small in the limit $N \to 0$ and so the low-frequency asymptotics may be obtained in a relatively straightforward manner. Cole (1977) obtained a similar integral equation by a conformal mapping technique. He showed that $[\phi]$ satisfies (5.4) with

$$\rho^*(\theta_t,\theta_q) = -(2/\pi)\,[\ln|\sin\tfrac{1}{2}\gamma\cos\tfrac{1}{2}\gamma\sin\tfrac{1}{2}(\theta_t-\theta_q)| - \ln|\sin^2\tfrac{1}{2}\gamma\cos\tfrac{1}{2}(\theta_t-\theta_q) - \sin\tfrac{1}{2}\theta_t\sin\tfrac{1}{2}\theta_q + S(\theta_t)\,S(\theta_q)|], \tag{6.11}$$

where

$$S(\theta) = \surd(\sin^2\tfrac{1}{2}\gamma - \sin^2\tfrac{1}{2}\theta). \tag{6.12}$$

The theory in §5 shows that our equation and Cole's are in fact equivalent. Cole gives several terms in the low frequency expansion.

7. Scattering by arcs of general shape

In the previous section we considered the only two cases for which $K_0(p,q) \equiv 0$. Here we will consider the case of an arc given by $y = f(x)$, where f is a twice continuously differentiable function with a single-valued inverse f^{-1}, see figure 3.

(a) *Simplification of the kernel*

From (4.1) and (4.6) the expression for the kernel in our Fredholm integral equation is

$$\hat{K}(p,q) = \frac{\partial}{\partial n_p}\int_L \rho^*(t,q)\,\frac{\partial G}{\partial n_t}(t,p)\,\mathrm{d}s_t, \tag{7.1}$$

and it is a relatively simple matter to show that

$$\hat{K}(p,q) = -\frac{\partial}{\partial s_p}\int_L \rho^*(t,q)\,\frac{\partial G}{\partial s_t}(t,p)\,\mathrm{d}s_t + N^2\int_L \rho^*(t,q)\,\boldsymbol{n}_p\cdot\boldsymbol{n}_t\,G(t,p)\,\mathrm{d}s_t. \tag{7.2}$$

Performing an integration by parts, we obtain

$$\hat{K}(p,q) = \frac{\partial}{\partial s_p}\int_L \frac{\partial\rho^*}{\partial s_t}(t,q)\,G(t,p)\,\mathrm{d}s_t + N^2\int_L \rho^*(t,q)\,\boldsymbol{n}_p\cdot\boldsymbol{n}_t\,G(t,p)\,\mathrm{d}s_t, \tag{7.3}$$

where the first integral must be interpreted as a Cauchy principle value. If $p \neq q$, the order of integration and differentiation may be interchanged so that

$$\hat{K}(p,q) = \int_L \frac{\partial G}{\partial s_p}(t,p)\frac{\partial \rho^*}{\partial s_t}(t,q)\,\mathrm{d}s_t + N^2 \int_L \boldsymbol{n}_p \cdot \boldsymbol{n}_t\, \rho^*(t,q)\, G(t,p)\,\mathrm{d}s_t \tag{7.4}$$

and the Cauchy Principle Value is taken at $t = p$ and $t = q$.

For an arbitrary surface, the derivatives may be written as

$$\frac{\partial}{\partial n_t} = \frac{1}{\sqrt{(1+f'(x_t)^2)}}\left\{\frac{\partial}{\partial y_t} - f'(x_t)\frac{\partial}{\partial x_t}\right\}. \tag{7.5}$$

Further, we may transform the surface integrals in (7.4) into integrals along the 'mean plane', i.e. $y = 0$, $|x| \leqslant 1$. This is simply achieved by writing

$$\mathrm{d}s_t = \sqrt{(1+f'(x_t)^2)}\,\mathrm{d}x_t.$$

Expanding the derivatives in (7.4) we obtain the following expression for the kernel:

$$K(x_p, x_q) = -\tfrac{1}{4}\mathrm{i}N^2 \int_{-1}^{1} \frac{\sqrt{(1+f'(x_t)^2)}}{\sqrt{(1+f'(x_p)^2)}}\frac{\partial \rho^*}{\partial s_t}(t,q)\, F_1(x_p, x_t)\,\mathrm{d}x_t + \tfrac{1}{4}\mathrm{i}N^2 \int_{-1}^{1} \rho^*(t,q)\, F_2(x_p, x_t)\,\mathrm{d}x_t, \tag{7.6}$$

where

$$F_1(x_p, x_t) = \left[\frac{(x_p - x_t) + f'(x_p)\,(f(x_p) - f(x_t))}{N\sqrt{[(x_p-x_t)^2 + (f(x_p)-f(x_t))^2]}}\right] \times H_1^{(1)}(N\sqrt{[(x_p-x_t)^2 + (f(x_t)-f(x_p))^2]}) \tag{7.7}$$

and

$$F_2(x_p, x_t) = \frac{f'(x_t)f'(x_p)+1}{\sqrt{(1+f'(x_p)^2)}} H_0^{(1)}(N\sqrt{[(x_p-x_t)^2 + (f(x_t)-f(x_p))^2]}). \tag{7.8}$$

(*b*) *Choice for* $\rho^*(t,q)$

From §4, we have seen that we may choose $\rho^*(t,q)$ to be

$$\rho^*(t,q) = -(2/\pi)\{\ln|s_t - s_q| - \ln|s_0^2 - s_q s_t + \sqrt{(s_0^2-s_q^2)}\sqrt{(s_0^2-s_t^2)}| + \ln|s_0|\} + \sigma(t,q).$$

Here we exploit the arbitrariness of $\sigma(t,q)$ to obtain the simplest form for the integral equation. In order to motivate our choice, we examine the behaviour of the integrands in (7.6). From (7.7) and (7.8) we can see that

$$F_1(x_p, x_t) = -(2\mathrm{i}/\pi N^2(x_p - x_t)) + O(1), \quad x_p \to x_t$$

and

$$F_2(x_p, x_t) = (2\mathrm{i}/\pi)\sqrt{(1+f'(x_p)^2)}\ln|x_t - x_p| + O(1),$$

which are independent of $f(x_t)$, i.e. they are largely insensitive, to this order, to the geometry of the scatterer. It therefore seems reasonable to choose $\sigma(t,q)$ such that

$$\rho^*(t,q) \equiv \rho(x_t, x_q). \tag{7.9}$$

The simplicity of this solution is exemplified by the identity

$$\sqrt{(1+f'(x_t)^2)}\frac{\partial \rho^*}{\partial s_t}(t,q) = \frac{\partial \rho}{\partial x_t}(x_t, x_q),$$

which, in particular, means that we may integrate the logarithmic singularities in (7.7), and $\rho(x_t, x_q)$ *is independent of the form of the surface*. This is clearly a computationally more convenient form than the fundamental CGF, given by (4.2) for this special class of scatterers.

We note that the most singular term in (7.6) gives zero contribution, since, by contour integration,

$$\int_{-1}^{1} \frac{1}{(x_p - x_t)} \frac{\partial \rho}{\partial x_t}(x_t, x_q) \mathrm{d}x_t = \surd(1-x_q^2) \int_{-1}^{1} \frac{\mathrm{d}x_t}{\surd(1-x_t^2)(x_p-x_t)(x_t-x_q)} = 0, \tag{7.10}$$

and so, after an integration by parts, the final form of the kernel is

$$K(x_p, x_q) = \frac{\mathrm{i}N^2}{4\surd(1+f'(x_p)^2)} \int_{-1}^{1} \rho(x_t, x_q) \left\{ a(x_t, x_p) H_0^{(1)}(N\surd(x^2+s^2)) - b(x_t, x_p) \frac{H_1^{(1)}(N\surd(x^2+s^2))}{N\surd(x^2+s^2)} + \frac{2\mathrm{i}}{\pi N^2 (x_t - x_p)^2} \right\} \mathrm{d}x_t, \tag{7.11}$$

where

$$\left. \begin{aligned} a(x_t, x_p) &= f'(x_t) f'(x_p) + 1 - (x + f'(x_p)s)(x + f'(x_t)s)/(x^2+s^2), \\ b(x_t, x_p) &= f'(x_t) f'(x_p) + 1 - 2(x + f'(x_p)s)(x + f'(x_t)s)/(x^2+s^2), \end{aligned} \right\} \tag{7.12}$$

and $x = x_p - x_t$ and $s = f(x_p) - f(x_t)$. Thus we have arrived at a remarkably simple expression for the kernel of our integral equation for the class of scattering surfaces defined above.

Finally in this section, we consider the behaviour of the crack opening displacement (COD). From (5.4) we can see that

$$\mu(q) = \int_L \rho(t, q) \frac{\partial \phi_0}{\partial n_t} \mathrm{d}s_t + \int_L \hat{K}(p, q) \mu(p) \mathrm{d}s_p. \tag{7.13}$$

Also, using (7.11) and the special form chosen for $\rho(t, q)$, we can deduce that the COD has the following behaviour at the endpoints

$$\mu(q) = O((1 \pm q)^{\frac{1}{2}}), \quad q \to \mp 1, \tag{7.14}$$

which conforms with the edge condition (2.3) and the ubiquitous stress singularity at the edge of a crack.

8. Numerical analysis and results

The form of the kernel (7.11) is

$$K(x_p, x_q) = \int_{-1}^{1} \rho(x_t, x_q) F(x_t, x_p) \mathrm{d}x_t, \tag{8.1}$$

where $F(x_t, x_p)$ is a linear combination of the Hankel functions $H_0^{(1)}(Nz)$ and $H_1^{(1)}(Nz)$, where $z = \surd[(x_t - x_p)^2 + (f(x_t) - f(x_p))^2]$. This integral cannot be performed explicitly and so it is necessary to devise a scheme for evaluating it efficiently. From (7.11) $F(x_t, x_p)$ may be written as

$$F(x_t, x_p) = A(x_p) \ln|x_t - x_p| + B(x_p)(x_t - x_p) \ln|x_t - x_p| + C(x_p)(x_t - x_p)^2 \ln|x_t - x_p| + F_0(x_t, x_p), \tag{8.2}$$

where $F_0(x_t, x_p)$ is a twice continuously differentiable function and $A(x_p)$, $B(x_p)$ and $C(x_p)$ are given by

$$A(x_p) = (\mathrm{i}/\pi)\,(1+f'(x_p)^2),$$
$$B(x_p) = (\mathrm{i}/\pi) f'(x_p) f''(x_p),$$
$$C(x_p) = -(\mathrm{i}N^2/8\pi)\,(1+f'(x_p)^2) + (\mathrm{i}/2\pi) f'(x_p) f'''(x_p).$$

It is to be noted that the expression for $C(x_p)$ contains the third derivative of the surface profile. This extra term arises from the method of solution i.e. the expansion of the kernel into a Taylor series. The theory presented in §2 through §7 only requires that the scattering profile be twice continuously differentiable.

In Appendix E we show that the integrals which multiply $A(x_p)$, $B(x_p)$ and $C(x_p)$ may be performed explicitly; thus we have

$$\begin{aligned} K(x_p, x_q) = {} & 2A(x_p)\left\{2|x_p - x_q| \arctan \sqrt{\left(\frac{1 \pm x_q}{1 \mp x_q}\right)} - (1+\ln 2)\sqrt{(1-x_q^2)}\right\} \\ & + B(x_p)\left\{-2(x_p - x_q)\,|x_p - x_q| \arctan \sqrt{\left(\frac{1 \pm x_q}{1 \mp x_q}\right)} \right. \\ & \left. - \tfrac{1}{2}[x_q + 2x_q \ln 2 - 4x_p \ln 2]\sqrt{(1-x_q^2)}\right\} \\ & + C(x_p)\left\{\tfrac{4}{3}(x_p - x_q)^2\,|x_p - x_q| \arctan \sqrt{\left(\frac{1 \pm x_q}{1 \mp x_q}\right)} + \sqrt{(1-x_q^2)} \right. \\ & \left. \times [x_p^2 - 2x_p^2 \ln 2 + 2x_p x_q \ln 2 - \tfrac{2}{9}x_q^2[1+3\ln 2] + \tfrac{1}{18}[1 - 6\ln 2]]\right\} \\ & + \int_{-1}^{1} F_0(x_t, x_p)\,\rho(x_t, x_q)\,\mathrm{d}x_t, \end{aligned} \tag{8.3}$$

where $\pm$ corresponds to $x_p > x_q$ and $x_p < x_q$ respectively. It now remains to find a method for efficiently evaluating the remaining integral.

The obvious numerical scheme for the last term in (8.3) is a product rule, in which case we need to choose an appropriate set of basis functions for approximating F_0, regarded as a function of x_t, for each value of x_p. Since we have chosen F_0 to be C^2 then a number of alternatives present themselves. The most natural approach is to use Chebyshev polynomials because

$$\int_{-1}^{1} \rho(x_t, x_q)\, T_n(x_t)\,\mathrm{d}x_t = \sqrt{(1-x_q^2)}\left[\frac{U_n(x_q)}{n+1} - \frac{U_{n-2}(x_q)}{n-1}\right], \quad n > 1, \tag{8.4}$$

which is easily established using the result (22.13.4) in Abramowitz & Stegun (1968). Here the functions $T_n(x)$, $U_n(x)$ are the Chebyshev polynomials of the first and second kind. This leads to the approximation

$$\begin{aligned} \int_{-1}^{1} \rho(x_t, x_q)\, F_0(x_t, x_p)\,\mathrm{d}x_t = {} & \sqrt{(1-x_q^2)}\,\{2c_0(x_p) + x_q c_1(x_p) \\ & + \sum_{n=2}^{N} c_n(x_p)\left[\frac{U_n(x_q)}{n+1} - \frac{U_{n-2}(x_q)}{n-1}\right]\right\}, \end{aligned} \tag{8.5}$$

and effectively reduces the integration to the determination of the Chebyshev coefficients $c_n(x_p)$. This scheme is fast and has the added advantage that $\sqrt{(1-x_q^2)}$

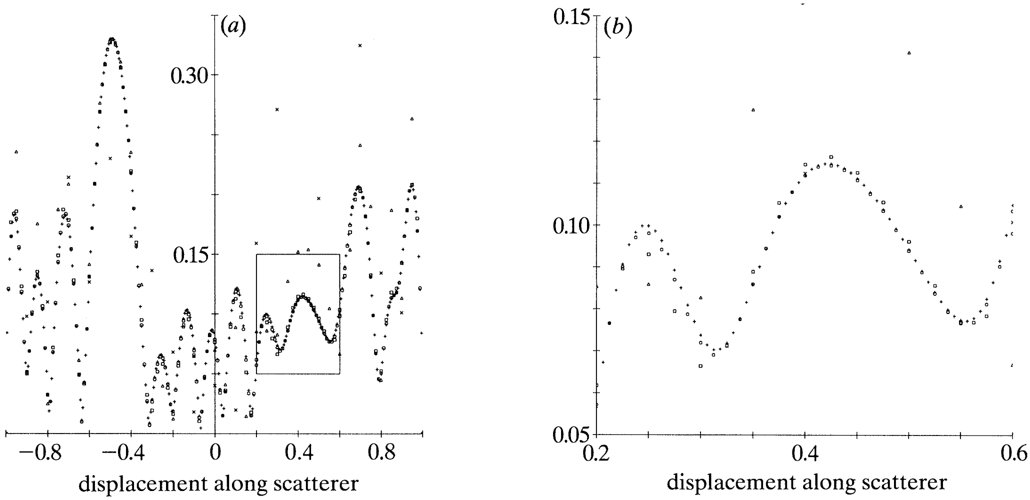

Figure 4. Stability of the results obtained for the COD for $N = 10$.

is an explicit factor. This in turn enables us to explicitly display the singular behaviour of $\mu(x_q)$ in the integral equation (7.13). Thus we write

$$\mu(x) = \sqrt{(1-x^2)}\, u(x), \quad \text{say}, \tag{8.6}$$

so that $u(x_q)$ satisfies

$$\sqrt{(1-x_q^2)}\, u(x_q) - \int_{-1}^{1} \sqrt{(1-x_p^2)}\, u(x_p) \sqrt{(1+f'(x_p)^2)}\, K(x_p, x_q)\, \mathrm{d}x_p = \mu_0(x_q). \tag{8.7}$$

We further note that

$$\lim_{x_q \to \pm 1} \frac{1}{\sqrt{(1-x_q^2)}} \arctan \sqrt{\left(\frac{1 \pm x_q}{1 \mp x_q}\right)} = \tfrac{1}{2} \tag{8.8}$$

and so we can see that our kernel (8.3) is continuous on $(x_q, x_p) \in [-1,1] \times [-1,1]$. This important result means that we have a large degree of flexibility in choosing a numerical scheme for the solution of (8.7), cf. Baker (1977). The only remaining difficulty is that $K(x_p, x_q)$ has a discontinuity in its first derivative on the diagonal $x_p = x_q$, but this may easily be accommodated in our choice of quadrature rule. Here we have chosen to adopt what is probably the simplest and fastest numerical method for solving (8.7), namely the Fox–Goodwin algorithm (Fox & Goodwin 1953). This uses the trapezium rule and an iterative refinement by the Gregory correction formula; in our case this has to be applied on the piecewise continuous parts of the range of integration.

Finally, we note that unlike boundary element methods of solution the shape of the boundary may be included 'exactly' and so the only sources of numerical truncation error in our method are in the approximation (8.5) and in our solution scheme for (8.7). It is well known that for Fredholm integral equations of the second kind with compact operators it is relatively easy to control the latter (cf. Baker 1977, ch. 4).

The first set of results that we present are those for the scattering surface $y = \sin \pi x$. Figure 4*a*, *b* illustrates the stability of the results obtained for the COD. Here we have used 21, 41, 81, 161 and 321 points in the Fox–Goodwin algorithm, with N,

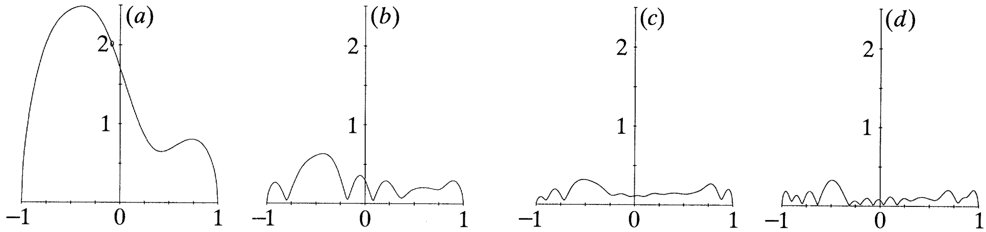

Figure 5. Illustration of the variation of the COD with wavenumber. (*a*) $N = 1$; (*b*) $N = 4$; (*c*) $N = 7$; (*d*) $N = 10$.

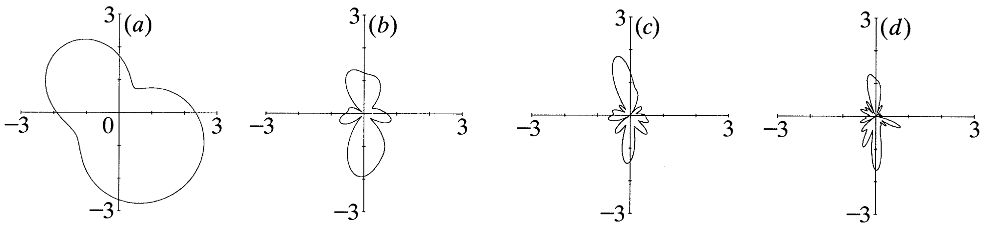

Figure 6. Polar plot of the scattered fields associated with figure 5. (*a*) $N = 1$; (*b*) $N = 4$; (*c*) $N = 7$; (*d*) $N = 10$.

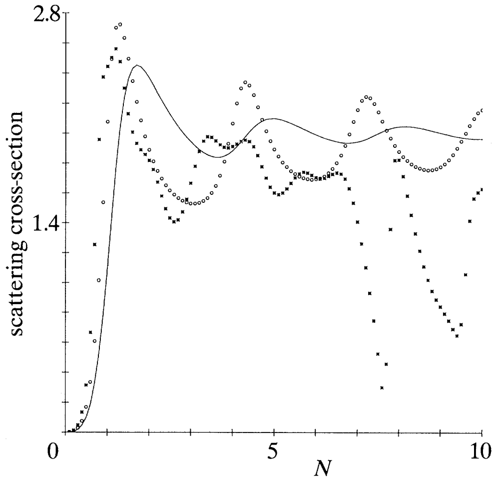

Figure 7. Scattering cross-sections.

the non-dimensional wavenumber, equal to 10. An enlargement of a small portion of figure 4*a* is shown in figure 4*b*. We can clearly see that the results are converging quite rapidly, even for large values of N.

Figure 5 shows examples of the COD as the wavenumber varies. In particular, we have chosen $N = 1$, 4, 7 and 10. Figure 6 shows a polar plot of the scattered fields associated with each wavenumber.

Finally for our 'benchmark' scatterer, we present the results obtained for the scattering cross-section, defined as the scattered field in the forward direction. For an incident plane wave of the form

$$\phi_0 = \exp\left[\mathrm{i}N(x\sin\alpha + y\cos\alpha)\right],$$

the scattering cross-section is found by substituting $\theta = \frac{1}{2}\pi - \alpha$ into (B 2). For

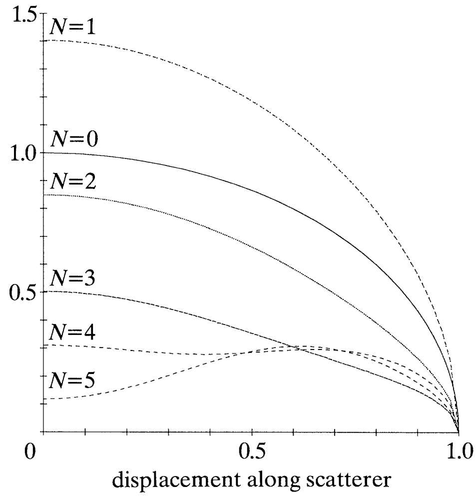

Figure 8. Variation of the COD with wavenumber for a flat scatterer.

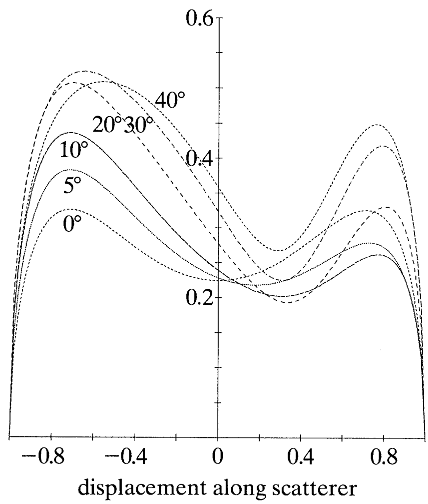

Figure 9. Variation of the COD with angle of incidence for a semi-circular arc ($N = 3$).

comparison, we have plotted the results obtained when the scatterer is flat and when we have a semi-circular arc of the form

$$y = \surd(a^2 - x^2),$$

where $a = 1.1$. We note that we are unable to choose $a = 1.0$, since the theory presented in this section requires a 1:1 correspondence between the scattering surface and the plane $y = 0$.

Having presented various results for one specific surface, we next examine the CODs obtained from other scattering profiles. Figure 8 shows the results obtained when the scattering surface is flat. These are in excellent agreement with those presented by Mal (1970). Figure 9 shows the variation in the COD as the angle of incidence varies for the semi-circular arc with $N = 3$. Figure 10a, b illustrate the CODs obtained from surfaces of the form

$$y = \sin(\pi n x)/10, \quad n = 1, \ldots, 4,$$

for incident wavenumbers of $N = 3$ and 6.

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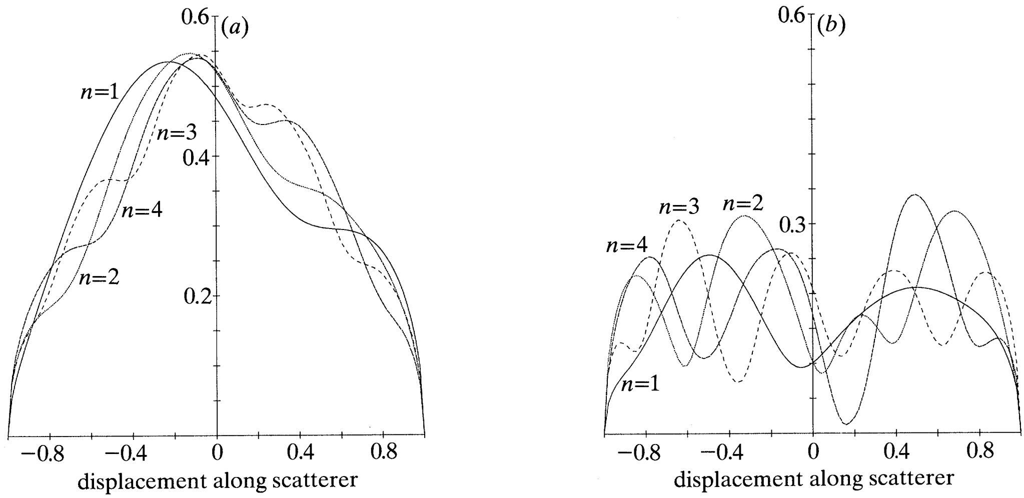

Figure 10. Examples of the COD for sinusoidal surfaces. (*a*) $N = 3$; (*b*) $N = 6$.

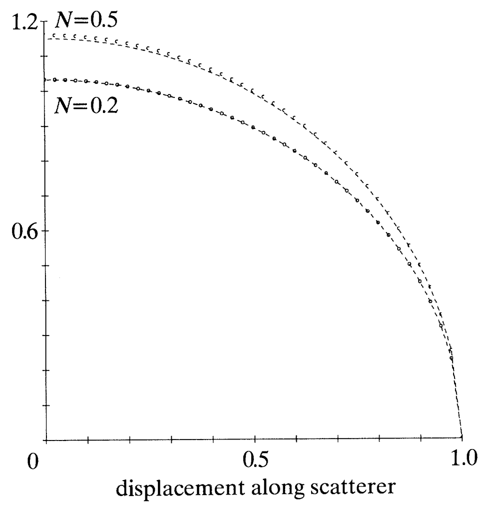

Figure 11. Comparison of low frequency asymptotic (----) results with numerical (oooo) results.

Finally we compare the results obtained using our numerical scheme with the low frequency asymptotic result of §6*a*. Results are presented for $N = 0.2$ and $N = 0.5$. As can be seen in figure 11, there is good agreement, as would be expected.

9. Conclusion

In this paper we have developed an exact rigorous theory for determining the potential discontinuity across an *arbitrary* finite open 'rigid' arc irradiated by scalar waves satisfying Helmholtz's equation. The problem has been reduced to a certain Fredholm integral equation of the second kind with a continuous kernel. In the case when the arc is twice differentiable and there exists a 1:1 mapping from the arc to a straight line segment, we have provided a fast and accurate numerical scheme for solving this integral equation. Our method is *rigorous* in that we have proved the existence and uniqueness of the solution of the original boundary value problem and *exact* by virtue of the fact that we can, in principle, evaluate the solution to any required accuracy. In addition the method used here has the following practical advantages.

1. In principle, there are no restrictions on the size of the wavenumber N or the 'surface heights and gradients', $f(x)$, $f'(x)$.

2. The method explicitly displays the known singularity in the potential discontinuity $\mu(x)$; other methods, such as the boundary element technique for the hypersingular equation, require the use of special elements to correctly model this singular behaviour.

3. There is an abundance of simple numerical schemes for solving Fredholm integral equations of the second kind. At low to moderate frequencies ($N = O(1)$) our equation is as good numerically as an explicit formula for $\mu(x)$.

4. The second kind integral (7.13) equation provides a rigorous interpolation formula. Thus it is possible to calculate the $\mu(x)$ at any point irrespective of the solution method. This of course provides stark contrast with numerical methods based on the solution of a *first kind* equation such as (3.3).

The general approach to the scattering by open surfaces and arcs by using the discontinuous Green functions (CGF) has a number of important applications in classical physics. In a future article it will be shown how to generalize the theory provided here to the problem of scattering of elastic waves by an arbitrary shaped crack in two dimensions. In a further paper we shall show how the discontinuous Green functions may be used to solve the corresponding problems for piecewise continuous arcs and arcs on which impedance boundary conditions are applied.

Appendix A. Proof of Theorem 1

Let $\mathscr{D}_\epsilon$ denote the domain interior to the circle $r = R$, denoted by $\mathscr{C}_R$, and excluding the points on L and the domain $\mathscr{G}_\epsilon$ interior to the circles $\mathscr{C}_\epsilon$ of radius ϵ centred at E^- and E^+ (see figure 1). Then it follows from the divergence theorem that

$$\int_{\mathscr{D}_\epsilon} |\mathrm{grad}\,\phi|^2\,\mathrm{d}S = \mathrm{Re}\,(\bar{k}^2)\int_{\mathscr{D}_\epsilon} |\phi|^2\,\mathrm{d}S + \mathrm{Re}\int_{\mathscr{C}} \phi\frac{\partial\bar{\phi}}{\partial n}\,\mathrm{d}s \tag{A 1}$$

and

$$\mathrm{Im}\,(\bar{k}^2)\int_{\mathscr{D}_\epsilon} |\phi|^2\,\mathrm{d}S + \mathrm{Im}\int_{\mathscr{C}} \phi\frac{\partial\bar{\phi}}{\partial n}\,\mathrm{d}s = 0, \tag{A 2}$$

where $\mathscr{C} \equiv \mathscr{C}_R \cup L_+ \cup L_- \cup \mathscr{C}_\epsilon$ is the positively oriented boundary contour of $\mathscr{D}_\epsilon$, $\phi \in C^2(\mathscr{D}_\epsilon)$ and $\mathrm{grad}\,\phi$ is continuous from $\mathscr{C}$. Suppose now that ϕ is any solution of $\mathscr{S}(0)$ and define $f(\epsilon)$ by

$$f(\epsilon) = \int_{\mathscr{D}_\epsilon} |\mathrm{grad}\,\phi|^2\,\mathrm{d}S - \mathrm{Re}\,(\bar{k}^2)\int_{\mathscr{D}_\epsilon\cup\mathscr{G}_\epsilon} |\phi|^2\,\mathrm{d}S - \mathrm{Re}\int_{\mathscr{C}_R} \phi\frac{\partial\bar{\phi}}{\partial n}\,\mathrm{d}s, \tag{A 3}$$

then

$$f'(\epsilon) = -\int_{\mathscr{C}_\epsilon} |\mathrm{grad}\,\phi|^2\,\mathrm{d}s, \tag{A 4}$$

i.e. $f(\epsilon)$ is monotone non-increasing as ϵ increases. Combining (A 3) and (A 1) gives

$$f(\epsilon) = -\operatorname{Re}(\bar{k}^2)\int_{\mathscr{G}_\epsilon} |\phi|^2 \,\mathrm{d}S + \operatorname{Re}\int_{\mathscr{C}_\epsilon} \phi \frac{\partial \bar{\phi}}{\partial n}\,\mathrm{d}s \tag{A 5}$$

and hence using (2.3) and Schwartz's inequality we obtain

$$|f(\epsilon)| \leqslant |\operatorname{Re}(\bar{k}^2)|\, 2\pi\epsilon^2 M^2 + \left\{\int_{\mathscr{C}_\epsilon} |\phi|^2\,\mathrm{d}s\right\}^{\frac{1}{2}} \left\{\int_{\mathscr{C}_\epsilon} \left|\frac{\partial\phi}{\partial n}\right|^2 \mathrm{d}s\right\}^{\frac{1}{2}}$$

$$\leqslant |\operatorname{Re}(\bar{k}^2)|\, 2\pi\epsilon^2 M^2 + 2M\sqrt{(\pi\epsilon)}\left\{\int_{\mathscr{C}_\epsilon} |\operatorname{grad}\phi|^2\,\mathrm{d}s\right\}^{\frac{1}{2}},$$

i.e.
$$|f(\epsilon)| \leqslant b(\omega^2)\,\epsilon^2 + c\sqrt{(-\epsilon f'(\epsilon))}, \tag{A 6}$$

say, where b and c are positive and independent of ϵ. Now Knowles & Pucik (1973) have shown that the differential inequality (A 6) implies that $f(0^+) = 0$ and hence it follows from (A 5) that

$$\lim_{\epsilon\to 0} \operatorname{Re}\int_{\mathscr{C}_\epsilon} \phi\frac{\partial\bar{\phi}}{\partial n}\,\mathrm{d}s = 0 \tag{A 7}$$

and from (A 3) that there exists $M_0 > 0$ such that

$$\int_{\mathscr{G}_\epsilon} |\operatorname{grad}\phi|^2\,\mathrm{d}S < M_0. \tag{A 8}$$

Further, equation (A 2) may be written in the form

$$\operatorname{Im}\int_{\mathscr{C}_\epsilon} \phi\frac{\partial\bar{\phi}}{\partial n}\,\mathrm{d}s = -g(R,\epsilon),$$

where
$$g(R,\epsilon) = \operatorname{Im}(\bar{k}^2)\int_{\mathscr{D}_\epsilon} |\phi|^2\,\mathrm{d}S + \operatorname{Im}\int_{\mathscr{C}_R} \phi\frac{\partial\bar{\phi}}{\partial n}\,\mathrm{d}s.$$

Now, since $\operatorname{Im}(\bar{k}^2) < 0$, $g(R,\epsilon)$ is monotone non-increasing as ϵ increases, hence by (2.3) and Schwartz's inequality

$$|g(R,0^+)| \leqslant |g(R,\epsilon)| \leqslant 2M\sqrt{(\pi\epsilon)}\left\{\int_{\mathscr{C}_\epsilon} |\operatorname{grad}\phi|^2\,\mathrm{d}s\right\}^{\frac{1}{2}},$$

i.e.
$$\int_{\mathscr{C}_\epsilon} |\operatorname{grad}\phi|^2\,\mathrm{d}s \geqslant \frac{|g(R,0^+)|^2}{4M^2\pi\epsilon},$$

which contradicts (A 8) unless $g(R,0^+) \equiv 0$. On taking the limit as $R\to\infty$ and using the radiation condition (2.4) we conclude that

$$\operatorname{Im}(\bar{k}^2)\int_{D_0} |\phi|^2\,\mathrm{d}S + \operatorname{Im}(-\mathrm{i}\bar{k})\lim_{R\to\infty}\int_{\mathscr{C}_R} |\phi|^2\,\mathrm{d}s = 0. \tag{A 9}$$

It follows immediately that if $\operatorname{Im}(k^2) > 0$ then

$$\phi \equiv 0, \quad P \in D_0 \tag{A 10}$$

which is the required result. In the special case $\mathrm{Im}\,(k) = 0$, (A 9) reduces to

$$\lim_{R\to\infty} \int_{\mathscr{C}_R} |\phi|^2\, \mathrm{d}s = 0,$$

and (A 10) follows from a result due to Rellich (1943). □

Appendix B. Proof of Theorem 2

It is clear that (3.1) satisfies (2.1) for it is permissible to differentiate under the integral as many times as we like for $P \notin L$. Further, as $P \to \infty$, we may replace the Bessel function by its asymptotic expansion to obtain

$$\phi(P) = (2/\pi k r)^{\frac{1}{2}}\, \mathrm{e}^{\mathrm{i}(kr-\pi/4)}\, F(\theta) + o(r^{-\frac{1}{2}}), \tag{B 1}$$

uniformly in $0 \leqslant \theta < 2\pi$, where

$$F(\theta) = \tfrac{1}{4}\mathrm{i} \int_L \mu(s_q) \frac{\partial}{\partial n_q} \exp\left[-\mathrm{i}kr_q \cos(\theta - \theta_q)\right] \mathrm{d}s_q \tag{B 2}$$

and so (3.1) also satisfies (2.4). Next, we note that the power series (Watson 1944) for the Bessel function gives

$$G(P,Q) = \frac{1}{2\pi} \ln \frac{1}{R(P,Q)} + G_1(P,Q), \tag{B 3}$$

where $R(P,Q) = |\boldsymbol{r}(P) - \boldsymbol{r}(Q)|$,

$$G_1(P,Q) = \tfrac{1}{4}\mathrm{i} - (1/2\pi)\,(\gamma + \ln(\tfrac{1}{2}k)) + (k^2R^2/8\pi) \ln(\tfrac{1}{2}kR) + O(k^2R^2), \tag{B 4}$$

and γ is Euler's constant. It follows that if we set

$$\xi(s) = \mathrm{Re}\,(\mu(s)) \quad \text{and} \quad \eta(s) = \mathrm{Im}\,(\mu(s)),$$

then (3.1) may be written as

$$\phi(P) = \mathrm{Re}\frac{1}{2\pi\mathrm{i}} \int_L \frac{\xi(t)}{(t-z)}\,\mathrm{d}t + \mathrm{i}\,\mathrm{Re}\frac{1}{2\pi\mathrm{i}} \int_L \frac{\eta(t)}{(t-z)}\,\mathrm{d}t + \phi_1(P),$$

where

$$\phi_1(P) = \int_L \mu(s_q) \frac{\partial}{\partial n_q} G_1(P,q)\,\mathrm{d}s_q,$$

$z = r\,\mathrm{e}^{\mathrm{i}\theta}$ and $t = r_q\,\mathrm{e}^{\mathrm{i}\theta_q}$. Now it has been shown by Muskhelishvili (1953, ch. 2), that if $\rho(s)$ has properties $\mathscr{P}(L)$, then the Cauchy integral

$$\Phi(z) = \frac{1}{2\pi\mathrm{i}} \int_L \frac{\rho(t)}{(t-z)}\,\mathrm{d}t$$

is continuous at E^-, E^+ and from both sides of L and has continuous boundary values $\Phi^+(t_0)$, $\Phi^-(t_0)$ satisfying

$$\Phi^+(t_0) - \Phi^-(t_0) = \rho(t_0), \quad t_0 \in L.$$

We conclude that since $\phi_1(P)$ is clearly continuous in the whole plane, then $\phi(P)$ satisfies (2.3) and

$$[\phi(p)] = \phi^+(p) - \phi^-(p) = \mu(p), \tag{B 5}$$

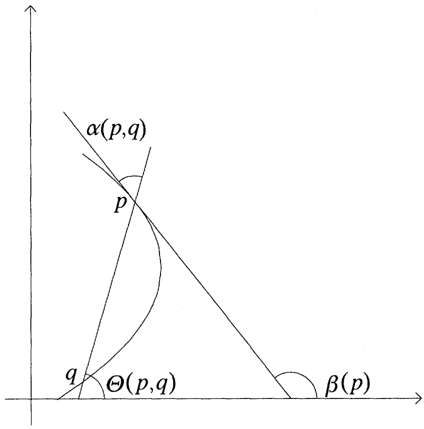

Figure 12. The angles α, β, Θ.

where $$\phi^{\pm}(p) = \phi(p), \quad p \in L^{\pm}.$$

Further, we may calculate the normal derivatives of (3.1) using the Cauchy–Riemann relations

$$\frac{\partial \Theta}{\partial s_q}(p,q) = \frac{\partial}{\partial n_q} \ln \frac{1}{R(p,q)}, \quad \frac{\partial \Theta}{\partial n_p}(p,q) = -\frac{\partial}{\partial s_p} \ln \frac{1}{R(p,q)}, \tag{B 6}$$

and an integration by parts, where $\Theta(p,q)$ is the angle between $\boldsymbol{r}(q)-\boldsymbol{r}(p)$ and $\mathrm{O}x$ (see figure 12); we find that

$$\frac{\partial \phi}{\partial n_p} = \frac{1}{2\pi} \frac{\partial}{\partial s_p} \int_L \mu'(s_q) \ln \frac{1}{R(p,q)} \mathrm{d}s_q + \frac{\partial \phi_1}{\partial n_p}. \tag{B 7}$$

The first term in this expression is the tangential derivative of a single layer harmonic potential and again Muskhelishvili (1953, ch. 2, §13) has shown that such an integral is continuous except at the ends E^- and E^+. Also, it follows from (B 4) that

$$\frac{\partial^2 G_1}{\partial n_p \partial n_q}(p,q) = O(\ln R(p,q))$$

as $p \to q$ and thus $\partial \phi_1 / \partial n_p$ is continuous on the whole of L. □

Appendix C. Proof of Theorem 3

It is easily shown using (B 4) that $\partial \hat{G}_1 / \partial R$ is continuous in the whole plane and thus we need only consider

$$\hat{G}_0(P,q) = \mathrm{Re} \frac{1}{2\pi \mathrm{i}} \int_L \frac{\rho(t,q)}{(t-z)} \mathrm{d}t. \tag{C 1}$$

An integration by parts gives

$$\frac{\partial \hat{G}_0}{\partial R} = \mathrm{Re} \left\{ \frac{\mathrm{e}^{\mathrm{i}\Theta(p,q)}}{2\pi \mathrm{i}} \int_L \frac{1}{(t-z)} \frac{\partial \rho}{\partial s_t} \mathrm{d}s_t \right\}. \tag{C 2}$$

But
$$\oint_L \frac{\partial \rho}{\partial s_t}\,\mathrm{d}s_t = 0 \tag{C 3}$$
and thus
$$\frac{\partial \hat{G}_0}{\partial R} = \mathrm{Re}\left\{\frac{\mathrm{e}^{\mathrm{i}\Theta(P,q)}}{2\pi\mathrm{i}}\oint_L \left\{\frac{1}{(t-z)} - \frac{1}{(q-z)}\right\}\frac{\partial \rho}{\partial s_t}\,\mathrm{d}s_t\right\}$$
$$= \mathrm{Re}\left\{\frac{\chi(z,q)}{2\pi\mathrm{i}R(P,q)}\right\}, \tag{C 4}$$
where
$$\chi(z,q) = \int_L \frac{\xi(t,q)}{(t-z)}\,\mathrm{d}t \tag{C 5}$$
and
$$\xi(t,q) = (t-q)\frac{\partial \rho}{\partial s_t}\mathrm{e}^{-\mathrm{i}\beta(s_t)}. \tag{C 6}$$
It follows from (4.2) that
$$\xi(t,q) = -\frac{2}{\pi}\frac{(t-q)}{(s_t-s_q)}\frac{\surd(s_0^2-s_q^2)}{\surd(s_0^2-s_t^2)}\mathrm{e}^{-\mathrm{i}\beta(s_t)}. \tag{C 7}$$
Now it is easily verified that the latter function is a Holder continuous function of s_t in the open interval $(-s_0, s_0)$ and hence from Muskhelishvili (1953, ch. 2, §22) it follows that if q is an interior point of L, then there exists positive constants C, ϵ and μ such that
$$|\chi(z,q)-\chi(q^{\pm},q)| < C|z-q|^{\mu}, \tag{C 8}$$
where $\mu \in (\epsilon, 1)$. But it follows from the Plemelj formulae and equations (C 6) and (C 7) that
$$\chi(q^{\pm},q) = \pm\pi\mathrm{i}\xi(q,q) + \oint_L \frac{\partial \rho}{\partial s_t}\,\mathrm{d}s_t = \mp 2\mathrm{i},$$
which proves (4.5).

To prove the second part of the theorem, consider
$$\hat{G}_0(P,q) = \frac{1}{2\pi}\int_L \rho(t,q)\frac{\partial}{\partial n_t}\ln R(P,t)\,\mathrm{d}s_t,$$
using the Cauchy relations (B 6) we obtain
$$\hat{K}_0(p,q) = \lim_{P\to p^{\pm}}\frac{\partial \hat{G}_0}{\partial n_p} = -\frac{1}{2\pi}\oint_L \frac{\cos\alpha(t,p)}{R(t,p)}\frac{\partial \rho}{\partial s_t}\,\mathrm{d}s_t, \tag{C 9}$$
where
$$\alpha(t,p) = \beta(p) - \Theta(t,p). \tag{C 10}$$
Hence
$$\frac{\partial \hat{G}_0}{\partial n_p} = -\frac{1}{2\pi}\mathrm{Re}\left\{\oint_L \frac{\mathrm{e}^{\mathrm{i}\beta(p)}}{(t-p)}\frac{\partial \rho}{\partial s_t}\,\mathrm{d}s_t\right\}$$
$$= \frac{\surd(s_0^2-s_q^2)}{\pi^2}\mathrm{Re}\left\{\oint_{-s_0}^{s_0}\frac{\mathrm{d}s_t}{(s_t-s_p)(s_t-s_q)\surd(s_0^2-s_t^2)} + \oint_L \frac{\eta(t,p)\,\mathrm{d}s_t}{(s_t-s_q)\surd(s_0^2-s_t^2)}\right\}, \tag{C 11}$$
where
$$\eta(t,p) = -\frac{1}{(s_t-s_p)} + \frac{\mathrm{e}^{\mathrm{i}\beta(p)}}{(t-p)}. \tag{C 12}$$

The first term in (C 11) may be shown to vanish identically using contour integration. The function $\eta(t,p)$ is evidently continuous on L and differentiable except at $t=p$. It follows that the second term in (C 11) has at worst a logarithmic singularity at $t=p$. Finally it is routine to show that

$$M_1(t,p) = O(\ln|t-p|), \tag{C 13}$$

as $t \to p$ which ensures the continuity of the integral in (4.7). This completes the proof of Theorem 3 save to note that, in general, $\hat{K}_0(p,q)$ may be expressed in the form (4.9) using the identity

$$\frac{\partial \Theta}{\partial s_t}(t,p) = \frac{\sin(\beta(s_t)-\Theta(t,p))}{R(t,p)}, \tag{C 14}$$

and (B 6). □

Appendix D. Proof of Theorem 4

We begin by first proving that every solution of (5.2) (and hence of (5.4)) has properties $\mathscr{P}(L)$. If $\partial\phi_0/\partial n_t$ is Holder continuous, then the right-hand side of (5.2) is also Holder continuous. It follows that every solution of (5.2) has the same property. Interchanging the order of integration then gives

$$\mu(q) = \int_L \rho(t,q)\, H(s_t)\, \mathrm{d}s_t$$

where
$$H(s_t) = \frac{1}{2\pi}\frac{\partial}{\partial n_t}\int_L \mu(p)\frac{\sin\alpha(t,p)}{R(t,p)}\mathrm{d}s_p + \int_L M_1(t,p)\,\mu(p)\,\mathrm{d}s_p + \frac{\partial \phi_0}{\partial n_t}. \tag{D 1}$$

Now, given that $\mu(p)$ is Holder continuous it is easily shown using (C 13) and (C 14) that $H(t)$ is also Holder continuous and hence

$$\frac{\mathrm{d}\mu(s_q)}{\mathrm{d}s_q} = -\frac{2}{\pi\surd(s_0^2-s_q^2)}\int_L \frac{\surd(s_0^2-s_t^2)}{(s_t-s_q)} H(s_t)\,\mathrm{d}s_t \tag{D 2}$$

exists and is Holder continuous on the interior of L (cf. Muskhelishvili 1953, ch. 2, §21).

Suppose now that we take any particular solution, μ^* say, of (5.2); then this may be substituted into (3.1) to generate a potential $\phi^*(P)$ which satisfies (2.1), (2.2) and (2.4). Further, if $\xi^* = \mathrm{Re}(\mu^*)$ and $\eta^* = \mathrm{Im}(\mu^*)$, then

$$\phi^*(P) = \mathrm{Re}\frac{1}{2\pi \mathrm{i}}\int_L \frac{\xi^*(t)}{(t-z)}\mathrm{d}t + \mathrm{i}\,\mathrm{Re}\frac{1}{2\pi\mathrm{i}}\int_L\frac{\eta^*(t)}{(t-z)}\mathrm{d}t + \int_L \mu^*(t)\frac{\partial G_1}{\partial n_t}\mathrm{d}s_t \tag{D 3}$$

and it may be shown using the methods of Muskhelishvili (1953, ch. 2, §16) that

$$\mathrm{grad}_p\,\phi^* = O(|z\mp q_0|^{-\frac{1}{2}}), \quad z\to\pm q_0. \tag{D 4}$$

Similarly
$$\mathrm{grad}_p\,\hat{G}(P,q) = O(|z\mp q_0|^{-\frac{1}{2}}), \quad z\to\pm q_0, \tag{D 5}$$

so that on applying Green's theorem to $\hat{G}$ and ϕ^* as in (5.1) we obtain

$$\mu^*(q) - \int_L \mu^*(p)\,\hat{K}(p,q)\,\mathrm{d}s_p = -\int_L \rho(t,q)\frac{\partial\phi^*}{\partial n_t}\mathrm{d}s_t, \quad q\in L, \tag{D 6}$$

since, by (B 5), $$[\phi^*(q)] = \mu^*(q), \tag{D 7}$$

and $\int_{\mathscr{C}_\epsilon} \to 0$ as $\epsilon \to 0$ by virtue of (2.3), (4.5) and (4.7). But μ^* satisfies (5.2) and hence (D 6) gives

$$\int_L \rho(t,q)\left(\frac{\partial \phi^*}{\partial n_t} + \frac{\partial \phi_0}{\partial n_t}\right) \mathrm{d}s_t = 0, \quad q \in L. \tag{D 8}$$

Differentiating this equation with respect to s_q gives

$$\frac{1}{\pi \surd(s_0^2 - s_q^2)} \int_L \frac{\surd(s_0^2 - s_t^2)}{(s_t - s_q)}\left(\frac{\partial \phi^*}{\partial n_t} + \frac{\partial \phi_0}{\partial n_t}\right) \mathrm{d}s_t = 0, \quad q \in L, \tag{D 9}$$

which implies that the analytic function $\Phi(z)$ defined by

$$\Phi(z) = \frac{1}{\pi \surd(s_0^2 - z^2)} \int_L \frac{\surd(s_0^2 - s_t^2)}{(s_t - z)}\left(\frac{\partial \phi^*}{\partial n_t} + \frac{\partial \phi_0}{\partial n_t}\right) \mathrm{d}t \tag{D 10}$$

has boundary values $$\Phi(q^\pm) = \frac{\partial \phi^*}{\partial n_q} + \frac{\partial \phi_0}{\partial n_q} \tag{D 11}$$

and is bounded at the ends $\pm s_0$. It follows that $\Phi(z)$ is an entire function and by Liouville's theorem it is identically zero; hence

$$\frac{\partial \phi^*}{\partial n_q} + \frac{\partial \phi_0}{\partial n_q} = 0, \tag{D 12}$$

i.e. ϕ^* satisfies the boundary condition (2.2). Now from Theorem 1 there is at most one such potential; we conclude that if there is a solution μ^* of (5.2) then it is also unique, but from Fredholm theory (Smithies 1962) uniqueness implies existence. This completes the proof of Theorem 4. □

Appendix E. Evaluation of certain integrals

In this appendix we evaluate the integrals that arose in §8. Firstly we consider

$$I_1(p,q) = \int_{-1}^{1} \sigma(t,q) \ln|t-p| \,\mathrm{d}t, \tag{E 1}$$

where $$\sigma(t,q) = \ln|t-q| - \ln|1 - tq + \surd(1-q^2)\surd(1-t^2)|. \tag{E 2}$$

An integration by parts gives

$$I_1(p,q) = \surd(1-q^2)\left\{\pi(1+\ln 2) - (q-p)\int_{-1}^{1} \frac{\ln|t-p|}{(t-q)\surd(1-t^2)} \mathrm{d}t\right\}, \tag{E 3}$$

where we have used $$\int_{-1}^{1} \frac{\ln|t-p|}{\surd(1-t^2)} \mathrm{d}t = -\pi \ln 2. \tag{E 4}$$

Next we consider

$$I_2(p,q) = \int_C \frac{\ln(z-p)}{(z-q)\surd(1-z^2)} \mathrm{d}z,$$

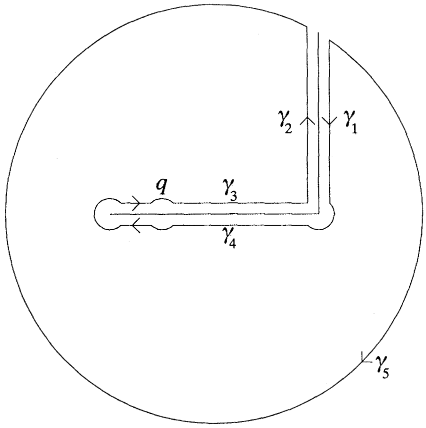

Figure 13. Contour of integration.

where C is the contour as shown in figure 13 and we have set $p = 1$. By the residue Theorem we see that

$$\int_{\gamma_3 \cup \gamma_4} \dots \mathrm{d}z = -\int_{\gamma_2 \cup \gamma_1} \dots \mathrm{d}z, \tag{E 5}$$

and that the right-hand side of (E 5) is in fact twice the required integral. Introducing $z = 1 + \mathrm{i}y$ and noting that due to the branch cut

$$\ln(z-1) = \ln|y| + \tfrac{1}{2}\mathrm{i}\pi \quad \text{on } \gamma_1,$$
$$\ln(z-1) = \ln|y| - \tfrac{3}{2}\mathrm{i}\pi \quad \text{on } \gamma_2,$$

we find that the resulting integral may be evaluated to give

$$I_2(1,q) = -\frac{2\pi}{\surd(1-q^2)} \arctan \sqrt{\left(\frac{1+q}{1-q}\right)}.$$

Repeating the above for $p = -1$, we obtain

$$I_2(-1,q) = \frac{2\pi}{\surd(1-q^2)} \arctan \sqrt{\left(\frac{1-q}{1+q}\right)},$$

and so we conclude that

$$I_2(p,q) = -2\pi \frac{\operatorname{sgn}(p-q)}{\surd(1-q^2)} \arctan \sqrt{\left(\frac{1 \pm q}{1 \mp q}\right)}, \tag{E 6}$$

where $\pm$ correspond to $p > q$ and $p < q$ respectively. Therefore

$$I_1(p,q) = \surd(1-q^2)\left\{\pi(1+\ln 2) - 2\pi \frac{|p-q|}{\surd(1-q^2)} \arctan \sqrt{\left(\frac{1 \pm q}{1 \mp q}\right)}\right\}, \tag{E 7}$$

as required.

The remaining integrals are evaluated in similar fashion using the following results:

$$\int_{-1}^{1} \frac{(t-p)\ln|t-p|}{\surd(1-t^2)} \mathrm{d}t = \pi(\ln 2 - 1)\, p$$

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and
$$\int_{-1}^{1} \frac{(t-p)^2 \ln|t-p|}{\sqrt{(1-t^2)}}\,\mathrm{d}t = \pi(\tfrac{3}{2}-\ln 2)\,p^2+\tfrac{1}{4}\pi(1-2\ln 2).$$

We find that

$$I_3(p,q) = \int_{-1}^{1} \sigma(t,q)\,(t-p)\ln|t-p|\,\mathrm{d}t$$
$$= \pi(p-q)\,|p-q| \arctan \sqrt{\left(\frac{1\pm q}{1\mp q}\right)} + \sqrt{(1-q^2)}\,\tfrac{1}{4}\pi\{q+2q\ln 2-4p\ln 2\}, \quad \text{(E 8)}$$

and
$$I_4(p,q) = \int_{-1}^{1} \sigma(t,q)\,(t-p)^2 \ln|t-p|\,\mathrm{d}t$$
$$= -\tfrac{2}{3}\pi(p-q)^2\,|p-q| \arctan \sqrt{\left(\frac{1\pm q}{1\mp q}\right)}$$
$$+\pi\sqrt{(1-q^2)}\,\{p^2\ln 2 - pq\ln 2+\tfrac{1}{3}q^2\ln 2+\tfrac{1}{6}\ln 2-\tfrac{1}{36}-\tfrac{1}{2}p^2+\tfrac{1}{9}q^2\}, \quad \text{(E 9)}$$

as required.

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